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Two Generalizations of the Projected Gradient Method for Convexly Constrained Inverse Problems —

Hybrid steepest descent method, Adaptive projected subgradient method

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Abstract In this paper, we present a brief review on the central results of two generalizations of a classical convex optimization technique named the *projected gradient method* [1, 2]. The 1st generalization has been made by extending the convex projection operator, used in the projected gradient method, to the (quasi-)nonexpansive mapping in a real Hilbert space. By this generalization, we deduce the *hybrid steepest descent method* [3–10] (see also [11]) that can minimize the convex cost function over the fixed point set of nonexpansive mapping [3–9, 11] (these results can also be interpreted as generalizations of fixed point iterations found for example in [12–15]) or, more generally, over the fixed point set of quasi-nonexpansive mapping [10]. Since (i) the solution set of wide range of convexly constrained inverse problems, for example in *signal processing and image reconstruction*, can be characterized as the fixed point set of certain nonexpansive mapping [5, 6, 9, 16–18], and (ii) subgradient projection operator and its variations are typical examples of quasi-nonexpansive mapping [10, 19], the hybrid steepest descent method has rich applications in broad range of mathematical sciences and engineerings. The 2nd generalization has been made for the *Polyak's subgradient algorithm* [20] that was originally developed as a version, of the *projected gradient method*, for unsmooth convex optimization problem with a fixed target value. By extending the *Polyak's subgradient algorithm* to the case where the convex cost function itself keeps changing in the whole process, we deduce the *adaptive projected subgradient method* [21–23] that can minimize asymptotically the sequence of unsmooth nonnegative convex cost functions. The *adaptive projected subgradient method* can serve as a unified guiding principle of a wide range of *set theoretic adaptive filtering schemes* [24–30] for nonstationary random processes. The great flexibilities in the choice of (quasi-)nonexpansive mapping as well as unsmooth convex cost functions in the proposed methods yield naturally inherently parallel structures (in the sense of [31]).

1 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|$. For a continuous convex function $\Phi:\mathcal{H}\to\mathbb{R}$, the *subdifferential* of Φ at $\forall y\in\mathcal{H}$, the set of all *subgradients* of Φ at y: $\partial\Phi(y):=\{g\in\mathcal{H}\mid\langle x-y,g\rangle+\Phi(y)\le\Phi(x),\forall x\in\mathcal{H}\}$ is nonempty. The convex function $\Phi:\mathcal{H}\to\mathbb{R}$ has a unique subgradient at $y\in\mathcal{H}$ if Φ is Gâteaux differentiable at y. This unique subgradient is nothing but the Gâteaux differential $\Phi'(y)$. A *fixed point* of a mapping $T:\mathcal{H}\to\mathcal{H}$ is a point $x\in\mathcal{H}$ such that $T(x)=x$. $Fix(T):=\{x\in\mathcal{H}\mid T(x)=x\}$ denotes the fixed point set of T. A mapping $T:\mathcal{H}\to\mathcal{H}$ is called (i) *strictly contractive* if $\|T(x)-T(y)\|\le\kappa\|x-y\|$ for some $\kappa\in(0,1)$ and all $x,y\in\mathcal{H}$ [The *Banach-Picard fixed point theorem* guarantees the unique existence of the fixed point, say $x_*\in Fix(T)$, of T and the strong convergence of $(T^n(x_0))_{n\ge0}$ to

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x_* for any $x_0 \in \mathcal{H}$.]; (ii) *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$, $\forall x, y \in \mathcal{H}$; (iii) *firmly nonexpansive* if $\|T(x) - T(y)\|^2 \leq \langle x - y, T(x) - T(y)\rangle$, $\forall x, y \in \mathcal{H}$ [32]; and (iv) *attracting nonexpansive* if T is nonexpansive with $Fix(T) \neq \emptyset$ and $\|T(x) - f\| < \|x - f\|$, $\forall f \in Fix(T)$ and $\forall x \notin Fix(T)$ [32]. Given a nonempty closed convex set $C \subset \mathcal{H}$, the mapping that assigns every point in $\mathcal{H}$ to its unique nearest point in C is called the *metric projection* or *convex projection* onto C and is denoted by P_C; i.e., $\|x - P_C(x)\| = d(x, C)$, where $d(x, C) := \inf_{y \in C} \|x - y\|$. P_C is firmly nonexpansive with $Fix(P_C) = C$ [32]. A mapping $T : \mathcal{H} \to \mathcal{H}$ is called *quasi-nonexpansive* if $\|T(x) - T(f)\| \leq \|x - f\|$, $\forall (x, f) \in \mathcal{H} \times Fix(T)$. In this paper, for simplicity, a mapping $T : \mathcal{H} \to \mathcal{H}$ is called *attracting quasi-nonexpansive* if $Fix(T) \neq \emptyset$ and $\|T(x) - f\| < \|x - f\|$, $\forall (x, f) \in Fix(T)^C \times Fix(T)$. Moreover, a mapping $T : \mathcal{H} \to \mathcal{H}$ is called *α-averaged quasi-nonexpansive* if there exists $\alpha \in (0, 1)$ and a quasi-nonexpansive mapping $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ such that $T = (1 - \alpha)I + \alpha\mathcal{N}$ (Note: $Fix(T) = Fix(\mathcal{N})$ holds automatically). In particular, 1/2-averaged quasi-nonexpansive mapping, which we specially call *firmly quasi-nonexpansive mapping.* Suppose that a continuous convex function $\Phi : \mathcal{H} \to \mathbb{R}$ satisfies $\mathrm{lev}_{\leq 0}\Phi := \{x \in \mathcal{H} \mid \Phi(x) \leq 0\} \neq \emptyset$. Then a mapping $T_{sp(\Phi)} : \mathcal{H} \to \mathcal{H}$ defined by

$$T_{sp(\Phi)} : x \mapsto \begin{cases} x - \frac{\Phi(x)}{\|g(x)\|^2} g(x) & \text{if } \Phi(x) > 0 \\ x & \text{if } \Phi(x) \leq 0, \end{cases} \tag{1}$$

where g is a selection of the subdifferential $\partial\Phi$, is called a *subgradient projection relative to* Φ [32].The mapping $T_{sp(\Phi)} : \mathcal{H} \to \mathcal{H}$ is *firmly quasi-nonexpansive* and satisfies $Fix(T_{sp(\Phi)}) = \mathrm{lev}_{\leq 0}\Phi$ (see for example [19]).

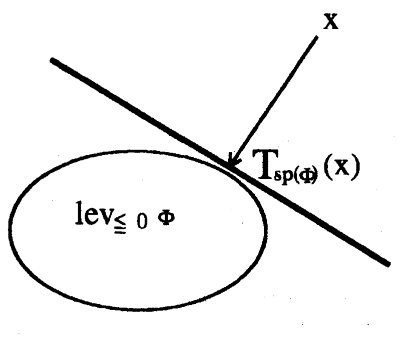

Figure 1: Subgradient projection relative to Φ

A mapping $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is called (i) *monotone* over $S \subset \mathcal{H}$ if $\langle \mathcal{F}(u) - \mathcal{F}(v), u - v\rangle \geq 0$, $\forall u, v \in S$. In particular, a mapping $\mathcal{F}$ which is monotone over $S \subset \mathcal{H}$ is called (ii) *paramonotone* over S if $\langle \mathcal{F}(u) - \mathcal{F}(v), u - v\rangle = 0 \Leftrightarrow \mathcal{F}(u) = \mathcal{F}(v)$, $\forall u, v \in S$ [34]; (iii) *uniformly monotone* over S if there exists a strictly monotone increasing continuous function $a : [0, \infty) \to [0, \infty)$, with $a(0) = 0$ and $a(t) \to \infty$ as $t \to \infty$, satisfying $\langle \mathcal{F}(u) - \mathcal{F}(v), u - v\rangle \geq a(\|u - v\|)\|u - v\|$ for all $u, v \in S$ [38]; (iv) *η-strongly monotone* over S if there exists $\eta > 0$ such that $\langle \mathcal{F}(u) - \mathcal{F}(v), u - v\rangle \geq \eta\|u - v\|^2$ for all $u, v \in S$ [38].

The *variational inequality problem* $VIP(\mathcal{F}, C)$ is defined as follows: given $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ which is monotone over a nonempty closed convex set $C \subset \mathcal{H}$, find $u^* \in C$ such that $\langle v - u^*, \mathcal{F}(u^*)\rangle \geq 0$, $\forall v \in C$. If a function $\Theta : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ is convex over a closed convex set C and Gâteaux differentiable with derivative Θ' over an open set $U \supset C$, then Θ' is paramonotone over C. For such a Θ, the set $\Gamma := \{u \in C \mid \Theta(u) = \inf \Theta(C)\}$ is nothing but the solution set of $VIP(\Theta', C)$ [33]. Given $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ which is monotone over a nonempty closed convex set C, $u^* \in C$ is a solution of VIP$(\mathcal{F}, C)$ if and only if $u^* \in Fix\,(P_C\,(I - \mu\mathcal{F}))$ for an arbitrarily fixed $\mu > 0$ (For related mathematical discussion in this section, the readers should consult, e.g., [6, 9, 19, 31–38]).

2 Hybrid Steepest Descent Method

Theorem 1 (Strong convergence for nonexpansive mapping [6, 9]**)** *Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping with $Fix\,(T) \neq \emptyset$. Suppose that a mapping $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is κ-Lipschitzian and η-strongly monotone over $T(\mathcal{H})$. Then, by using any sequence $(\lambda_n)_{n\geq 1} \subset [0, \infty)$ satisfying (W1) $\lim_{n\to+\infty} \lambda_n = 0$, (W2) $\sum_{n\geq 1} \lambda_n = +\infty$, (W3) $\sum_{n\geq 1} |\lambda_n - \lambda_{n+1}| < +\infty$ [or $(\lambda_n)_{n\geq 1} \subset (0, \infty)$ satisfying (L1) $\lim_{n\to+\infty} \lambda_n = 0$, (L2) $\sum_{n\geq 1} \lambda_n = +\infty$, (L3) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1})\lambda_{n+1}^{-2} = 0$], the sequence $(u_n)_{n\geq 0}$ generated, with arbitrary $u_0 \in \mathcal{H}$, by*

$$u_{n+1} := T(u_n) - \lambda_{n+1}\mathcal{F}\,(T(u_n)) \tag{2}$$

converges strongly to the uniquely existing solution of the VIP: find $u^* \in Fix(T)$ such that $\langle v - u^*, \mathcal{F}(u^*)\rangle \geq 0$, $\forall v \in Fix(T)$. (Note: The condition *(L3)* was relaxed recently to $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$ [11].) □

Theorem 1 is a generalization of a fixed point iteration [12–15] so called the *anchor method*:

$$u_{n+1} := \lambda_{n+1}a + (1 - \lambda_{n+1})T(u_n),$$

which converges strongly to $P_{Fix(T)}(a)$.

The *hybrid steepest descent method* (2) can be applied to more genaral monotone operators [7, 8] if $\dim(\mathcal{H}) < \infty$. Moreover, by the use of slowly changing sequence of nonexpansive mappings having same fixed point sets, a variation of the hybrid steepest descent method is gifted with notable robustness to the numerical errors possibly unavoidable in the iterative computations [9].

The next theorem shows that the hybrid steepest descent method can also be applied to the variational inequality problem over the fixed point set of *quasi-nonexpansive mappings.*

Definition 2 (Quasi-shrinking mapping[10]**)** *Suppose that $T : \mathcal{H} \to \mathcal{H}$ is quasi- nonexpansive with $Fix(T) \cap C \neq \emptyset$ for some closed convex set $C \subset \mathcal{H}$. Then $T : \mathcal{H} \to \mathcal{H}$ is called quasi-shrinking on $C(\subset \mathcal{H})$ if*

$$D : r \in [0, \infty) \mapsto \begin{cases} \displaystyle\inf_{u \in \rhd(Fix(T), r)\cap C} d(u, Fix(T)) - d(T(u), Fix(T)) \\ \qquad\qquad \text{if } \rhd(Fix(T), r) \cap C \neq \emptyset \\ \infty \qquad\qquad \text{otherwise} \end{cases}$$

satisfies $D(r) = 0 \Leftrightarrow r = 0$, where $\rhd(Fix(T), r) := \{x \in \mathcal{H} \mid d(x, Fix(T)) \geq r\}$. □

Proposition 3 [10] *Suppose that a continuous convex function $\Phi : \mathcal{H} \to \mathbb{R}$ has $\mathrm{lev}_{\leq 0} \neq \emptyset$ and bounded subdiferential $\partial\Phi : \mathcal{H} \to 2^{\mathcal{H}}$, i.e., $\partial\Phi$ maps bounded sets to bounded sets. Define $T_\alpha := (1 - \alpha)I + \alpha T_{sp(\Phi)}$ for $\alpha \in (0, 2)$ [hence $Fix(T_\alpha) = \mathrm{lev}_{\leq 0}\Phi$: see (1) for the definition of $T_{sp(\Phi)}$]. Then, we have the followings:*

(a) *If a selection of subgradient of Φ, say $\Phi' : \mathcal{H} \to \mathcal{H}$, is uniformly monotone over $\mathcal{H}$, then T_α is quasi-shrinking on any nonempty bounded closed convex set C satisfying $C \cap \mathrm{lev}_{\leq 0}\Phi \neq \emptyset$.*

(b) *Assume* $\dim(\mathcal{H}) < \infty$. *Then* T_α *is quasi-shrinking on any nonempty bounded closed convex set* $C(\subset \mathcal{H})$ *satisfying* $C \cap lev_{\leq 0}\Phi \neq \emptyset$. □

Theorem 4 *(***Strong convergence for quasi-shrinking mapping** [10]*) Suppose that* $T : \mathcal{H} \to \mathcal{H}$ *is a quasi-nonexpansive mapping with* $Fix(T) \neq \emptyset$. *Let* $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ *be* κ-*Lipschitzian and* η-*strongly monotone over* $T(\mathcal{H})$ *[Hence* $VIP(\mathcal{F}, Fix(T))$ *has its unique solution* $u^* \in Fix(T)$*]. Suppose also that there exists some* $(f, u_0) \in Fix(T) \times \mathcal{H}$ *for which* T *is quasi-shrinking on*

$$C_f(u_0) := \left\{ x \in \mathcal{H} \mid \|x - f\| \leq R_f := \max\left(\|u_0 - f\|, \frac{\|\mu\mathcal{F}(f)\|}{1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}} \right) \right\}.$$

Then for any $\mu \in (0, \frac{2\eta}{\kappa^2})$ *and any* $(\lambda_n)_{n\geq 1} \subset [0,1]$ *satisfying (H1)* $\lim_{n\to\infty} \lambda_n = 0$, *and (H2)* $\sum_{n\geq 1} \lambda_n = \infty$, *the sequence* $(u_n)_{n\geq 0}$, *generated by*

$$u_{n+1} := T(u_n) - \lambda_{n+1}\mu\mathcal{F}(T(u_n)),$$

converges strongly to u^*. □

If $\dim(\mathcal{H}) < \infty$, in a way similar to the discussions in [7–9], we can generalize Theorem 4 for application to more genaral monotone operators [10].

3 Adaptive Projected Subgradient Method

Theorem 5 (Adaptive Projected Subgradient Method [21, 22]**)** *Let* $\Theta_n : \mathcal{H} \to [0, \infty)$ $(\forall n \in \mathbb{N})$ *be a sequence of continuous convex functions and* $K \subset \mathcal{H}$ *a nonempty closed convex set. For an arbitrarily given* $u_0 \in K$, *the adaptive projected subgradient method produces a sequence* $(u_n)_{n\in\mathbb{N}} \subset K$ *by*

$$u_{n+1} := \begin{cases} P_K\left(u_n - \lambda_n \frac{\Theta_n(u_n)}{\|\Theta_n'(u_n)\|^2}\Theta_n'(u_n)\right) & \text{if } \Theta_n'(u_n) \neq 0, \\ u_n \text{ otherwise}, \end{cases}$$

where $\Theta_n'(u_n) \in \partial\Theta_n(u_n)$ *and* $0 \leq \lambda_n \leq 2$. *Then the sequence* $(u_n)_{n\in\mathbb{N}}$ *satisfies the followings.*

(a) *(Monotone approximation) Suppose that*

$$u_n \notin \Omega_n := \{u \in K \mid \Theta_n(u) = \Theta_n^*\} \neq \emptyset,$$

where $\Theta_n^* := \inf_{u\in K} \Theta_n(u)$. [1] *Then, by using* $\forall\lambda_n \in \left(0, 2\left(1 - \frac{\Theta_n^*}{\Theta_n(u_n)}\right)\right)$, *we have*

$$\forall u^{*(n)} \in \Omega_n, \|u_{n+1} - u^{*(n)}\| < \|u_n - u^{*(n)}\|.$$

(b) *(Boundedness, Asymptotic optimality) Suppose*

$$\exists N_0 \in \mathbb{N} \text{ s.t.} \begin{cases} \Theta_n^* = 0, \forall n \geq N_0 \text{ and} \\ \Omega := \bigcap_{n\geq N_0} \Omega_n \neq \emptyset. \end{cases} \tag{3}$$

Then $(u_n)_{n\in\mathbb{N}}$ *is bounded. Moreover if we specially use* $\forall\lambda_n \in [\varepsilon_1, 2 - \varepsilon_2] \subset (0, 2)$, *we have* $\lim_{n\to\infty} \Theta_n(u_n) = 0$ *provided that* $(\Theta_n'(u_n))_{n\in\mathbb{N}}$ *is bounded.*

[1] In this case, $\Theta_n(u_n) > \Theta_n^* \geq 0$.

(c) *(Strong convergence) Assume (3) and Ω has some relative interior w.r.t. a hyperplane $\Pi(\subset \mathcal{H})$, i.e., there exist $\widetilde{u} \in \Pi \cap \Omega$ and $\exists \varepsilon > 0$ satisfying $\{v \in \Pi \mid \|v - \widetilde{u}\| \leq \varepsilon\} \subset \Omega$. Then, by using $\forall \lambda_n \in [\varepsilon_1, 2 - \varepsilon_2] \subset (0, 2)$, $(u_n)_{n \in \mathbb{N}}$ converges strongly to some $\widehat{u} \in K$, i.e., $\lim_{n\to\infty} \|u_n - \widehat{u}\| = 0$. Moreover $\lim_{n\to\infty} \Theta_n(\widehat{u}) = 0$ if (i) $(\Theta_n'(u_n))_{n\in\mathbb{N}}$ is bounded and (ii) there exists bounded $(\Theta_n'(\widehat{u}))_{n\in\mathbb{N}}$ where $\Theta_n'(\widehat{u}) \in \partial\Theta_n(\widehat{u}), \forall n \in \mathbb{N}$.*

(d) *(A characterization of $\widehat{u}$) Assume the existence of some interior $\widetilde{u}$ of Ω, i.e., there exists $\varrho > 0$ satisfying $\{v \in \mathcal{H} \mid \|v - \widetilde{u}\| \leq \varrho\} \subset \Omega$. In addition to the conditions (i) and (ii) in (c), assume that there exists $\delta > 0$ satisfying*

$$\forall n \geq N_0, \forall u \in \Gamma \setminus (\mathrm{lev}_{\leq 0}\Theta_n), \exists \Theta_n'(u) \in \partial\Theta_n(u), \|\Theta_n'(u)\| \geq \delta,$$

where $\Gamma := \{(1-s)\widetilde{u} + s\hat{u} \in \mathcal{H} \mid s \in (0,1)\}$. Then, by using $\forall \lambda_n \in [\varepsilon_1, 2 - \varepsilon_2] \subset (0, 2)$, $\lim_{n\to\infty} u_n =: \widehat{u} \in \overline{\liminf_{n\to\infty} \Omega_n}$, where $\overline{\liminf_{n\to\infty} \Omega_n}$ stands for the closure of $\liminf_{n\to\infty} \Omega_n := \bigcup_{n\geq 0} \bigcap_{k\geq n} \Omega_k$. □

4 Concluding Remarks

In this paper, we briefly present central results on the *hybrid steepest descent method* and the *adaptive projected subgradient method* recently developed by our research group. For detailed mathematical discussions of the methods and their applications to inverse problems and signal processing problems, see [3–10, 16, 17, 21–23, 30] and references therein.

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